Equilibria und weiteres Heiteres

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Equilibria und weiteres Heiteres II-a

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August 15, 2011

Abstract

We investigate several technical and conceptual questions.

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1 Introduction

We present here various results, which may one day be published in a bigger paper, and which we wish to make already available to the community.

2 Countably many disjoint sets

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D. Makinson (personal communication).

We show here that, independent of the cardinality of the language, one can define only countably many inconsistent formulas.

The problem is due to D. Makinson (personal communication).

Example 2.1

There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge\{\neg p_j : j < i\} \wedge p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all i.

Let $i < i'$; we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

□

Fact 2.1

Any set X of consistent formulas with pairwise disjoint model sets is at most countable

Proof

Let such X be given.

(1) We may assume that X consists of conjunctions of propositional variables or their negations.

Proof: Rewrite all $\phi \in X$ as disjunctions of conjunctions ϕ_j. At least one of the conjunctions ϕ_j is consistent. Replace ϕ by one such ϕ_j. Consistency is preserved, as is pairwise disjointness.

(2) Let X be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in X with length i, i.e., a consistent conjunction of i many propositional variables or their negations, $i > 0$.

As the model sets for X are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each X_i is at most countable; we even show that each X_i is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let ϕ be p or $\neg p$. If ϕ' is not $\neg\phi$, then ϕ and ϕ' have a common model. So one must be p, the other $\neg p$. But these are all possibilities, so $card(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now X_i. Take arbitrary $\phi \in X_i$. Without loss of generality, let $\phi = p_1 \wedge \ldots \wedge p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, ϕ' must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$. Consider now $X_{i,k} := \{\phi' \in X_i : \phi' \text{ contains } \neg p_k\}$. Thus $X_i = \{\phi\} \cup \bigcup\{X_{i,k} : 1 \leq k \leq i\}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to X_{i-1}. So, by induction hypothesis, $card(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $card(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}$, $X_{i,k'}$ also have to be mutually inconsistent; our rough proof suffices.)

□

Note that the proof depends very little on logic. We needed normal forms, and used two truth values. Obviously, we can easily generalize to finitely many truth values.

3 Independence as ternary relation

3.1 Introduction

3.1.1 Independence

Independence is a central concept of reasoning.

In the context of non-monotonic logic and related areas like theory revision, it was perhaps first investigated formally by R. Parikh and co-authors, see e.g. [Par96], to obtain "local" conflict solution.

The present authors investigated its role for interpolation in preferential logics in [GS10], and showed connections to abstract multiplication of size.

Independence plays also a central role for a FOL treatment of preferential logics, where problems like the "dark haired Swedes" have to be treated. This is still subject of ongoing research.

J. Pearl investigated independence in graphs and pobabilistic reasoning, e.g. in [Pea88], also as a ternary relation, $\langle X \mid Y \mid Z\rangle$.

The aim of the present paper is to extend this abstract approach to the preferential situation. We should emphasize that this is only an abstract description of the independence relation, and thus not the same as independence for non-monotonic interpolation as examined in [GS10], where we *used* independence, essentially in the form of the multiplicative law $\mu(X \times Y) = \mu(X) \times \mu(Y)$, which says that the $\mu-$function preserves independence.

We have not investigated if an interesting form of interpolation results from some application of μ to situations described by $\langle X \mid Y \mid Z\rangle$, analogously to above application of μ to situations described by $\langle X \mid\mid Y\rangle$.

3.1.2 Overview

We will first discuss simple examples, to introduce the main ideas.

We then present the basic definitions formally, for probabilistic and set independence.

We then show basic results for set independence as a ternary relation, and turn to our main results, absence of finite characterization, and construction of new rules for this ternary relation.

3.1.3 Discussion of some simple examples

We consider here $X = Y = Z = W = \{0,1\}$ and their products. We will later generalize, but the main ideas stay the same. First, we look at $X \times Z$ (the Cartesian product of X with Z), then at $X \times Z \times W$, at $X \times Y \times Z$, finally at $X \times Y \times Z \times W$. Elements of these products, i.e., sequences, will be written for simplicity 00, 01, 10, etc., context will disambiguate. General sequences will often be written σ, τ, etc. We will also look at subsets of these products, like $\{00, 11\} \subseteq X \times Z$, and various probability measures on these products.

As a matter of fact, the main part of this article concerns subsets A of products $X_1 \times \ldots \times X_n$ and a suitable notion of independence for A, roughly, if we can write A as $A_1 \times \ldots \times A_m$. This will be made more precise and discussed in progressively more complicated cases in this section.

In the context of preferential structures, A is intended to be $\mu(X_1 \times \ldots \times X_n)$, the set of minimal models of $X_1 \times \ldots \times X_n$.

$X \times Z$ Let $P : X \times Z \to [0,1]$ be a (fixed) probability measure.

If $A \subseteq X \times Z$, we will set $P(A) := \Sigma\{P(\sigma) : \sigma \in A\}$.

If $A_x := \{\sigma \in X \times Z : \sigma(X) = x\}$, we will write $P(x)$ for $P(A_x)$, likewise $P(z)$ for $P(A_z)$, if $A_z := \{\sigma \in X \times Z : \sigma(Z) = z\}$. When these are ambiguous, we will e.g. write $A_{X=0}$ for $\{\sigma \in X \times Z : \sigma(X) = 0\}$, and $P(X = 0)$ for $P(A_{X=0})$, etc.

We say that X and Z are independent for this P iff for all $xz \in X \times Z$ $P(xz) = P(x) * P(z)$.

We write then $\langle X \mid\mid Z\rangle_P$, and call this and its variants probabilistic independence.

Example 3.1

(1)

$P(00) = P(01) = 1/6$, $P(10) = P(11) = 1/3$.

Then $P(X = 0) = 1/6 + 1/6 = 1/3$, and $P(X = 1) = 2/3$, $P(Z = 0) = 1/6 + 1/3 = 1/2$, and $P(Z = 1) = 1/2$, so $\langle X \mid\mid Z\rangle_P$.

(2)

$P(00) = P(11) = 1/3$, $P(01) = P(10) = 1/6$.

Then $P(X = 0) = P(X = 1) = P(Z = 0) = P(Z = 1) = 1/2$, but $P(00) = 1/3 \neq 1/2 * 1/2 = 1/4$, so $\neg\langle X \mid\mid Z\rangle_P$.

Definition 3.1

Consider now $\emptyset \neq A \subseteq X \times Z$ for general X, Z.

Define the following probability measure on $X \times Z$:

$$P_A(\sigma) := \begin{cases} \frac{1}{card(A)} & iff \quad \sigma \in A \\ \\ 0 & iff \quad \sigma \notin A \end{cases}$$

Example 3.2

(1)

$A := \{00, 01\}$,

then $P_A(00) = P_A(01) = 1/2$, $P_A(10) = P_A(11) = 0$, $P_A(X = 0) = 1$, $P_A(X = 1) = 0$, $P_A(Z = 0) = P_A(Z = 1) = 1/2$, and we have $\langle X \parallel Z\rangle_{P_A}$.

(2)

$A := \{00, 11\}$,

then $P_A(00) = P_A(11) = 1/2$, $P_A(01) = P_A(10) = 0$, $P_A(X = 0) = P_A(X = 1) = 1/2$, $P_A(Z = 0) = P_A(Z = 1) = 1/2$, but $P_A(00) = 1/2 \neq P_A(X = 0) * P_A(Z = 0) = 1/4$, and we have $\neg\langle X \parallel Z\rangle_{P_A}$.

(3)

$A := \{00, 01, 11\}$,

then $P_A(00) = P_A(01) = P_A(11) = 1/3$, $P_A(10) = 0$, $P_A(X = 0) = 2/3$, $P_A(X = 1) = 1/3$, $P_A(Z = 0) = 1/3$, $P_A(Z = 1) = 2/3$, but $P_A(00) = 1/3 \neq P_A(X = 0) * P_A(Z = 0) = 2/3 * 1/3 = 2/9$, and we have $\neg\langle X \parallel Z\rangle_{P_A}$.

Note that in (1) above, $A = \{0\} \times \{0, 1\}$, but neither in (2), nor in (3), A can be written as such a product. This is no coincidence, as we will see now.

More formally, we write $\langle X \parallel Z\rangle_A$ iff for all $\sigma\tau \in A$ there is $\rho \in A$ such that $\rho(X) = \sigma(X)$ and $\rho(Z) = \tau(Z)$, or, equivalently, that $A = \{\sigma(X) : \sigma \in A\} \times \{\sigma(Z) : \sigma \in A\}$, meaning that we can combine fragments of functions in A arbitrarily.

We call this and its variants set independence.

Fact 3.1

Consider above situation $X \times Z$. Then $\langle X \parallel Z\rangle_{P_A}$ iff $\langle X \parallel Z\rangle_A$.

Proof

"$\Rightarrow$":

$A \subseteq \{\sigma(X) : \sigma \in A\} \times \{\sigma(Z) : \sigma \in A\}$ is trivial. Suppose $P_A(x, z) = P_A(x) * P_A(z)$, but there are $\sigma, \tau \in A$, $\sigma(X)\tau(Z) \notin A$. Then $P_A(x), P_A(z) > 0$, but $P_A(x, z) = 0$, a contradiction.

"$\Leftarrow$":

Case 1: $P_A(x) = 0$, then $P_A(x, z) = 0$, and we are done. Likewise for $P_A(Z) = 0$.

Case 2: $P_A(x), P_A(z) > 0$.

By definition and prerequisite,

$P_A(x) = \frac{card\{\sigma\in A:\sigma(X)=x\}}{card(A)} = \frac{card\{\sigma(Z):\sigma\in A\}}{card(A)}$,

$P_A(z) = \frac{card\{\sigma\in A:\sigma(Z)=z\}}{card(A)} = \frac{card\{\sigma(X):\sigma\in A\}}{card(A)}$,

$P_A(x, z) = \frac{card\{\sigma\in A:\sigma(X)=x,\sigma(Z)=z\}}{card(A)} = \frac{1}{card(A)}$.

By prerequisite again, $card(A) = card\{\sigma(X) : \sigma \in A\} = card\{\sigma(Z) : \sigma \in A\}$, so $\frac{card\{\sigma(Z):\sigma\in A\}}{card(A)} * \frac{card\{\sigma(X):\sigma\in A\}}{card(A)}$ $= \frac{1}{card(A)}$

□

$X \times Z \times W$ Here, W will not be mentioned directly.

Let $P : X \times Z \times W \to [0, 1]$ be a probability measure.

Again, we say that X and Z are independent for P, $\langle X \parallel Z\rangle_P$, iff for all $x \in X$, $z \in Z$ $P(x, z) = P(x) * P(z)$.

Example 3.3

(1)

Let $P(000) = P(001) = P(010) = P(011) = 1/12$, $P(100) = P(101) = P(110) = P(111) = 1/6$, then X and Z are independent.

(2)

Let $P(100) = P(101) = P(010) = P(011) = 1/12$, $P(000) = P(001) = P(110) = P(111) = 1/6$, then $P(X = 0) = P(X = 1) = P(Z = 0) = P(Z = 1) = 1/2$, but $P(X = 0, Z = 0) = 1/3 \neq 1/2 * 1/2 = 1/4$, so $\neg\langle X \mid\mid Z\rangle_P$.

As above, we define P_A for $\emptyset \neq A \subseteq X \times Z \times W$.

Example 3.4

(1)

$A := \{000, 001, 010, 011\}$. Then $P_A(X = 0, Z = 0) = P_A(X = 0, Z = 1) = 1/2$, $P_A(X = 1, Z = 0) = P_A(X = 1, Z = 1) = 0$, $P_A(X = 0) = 1$, $P_A(X = 1) = 0$, $P_A(Z = 0) = P_A(Z = 1) = 1/2$, so X and Z are independent.

(2)

For $A := \{000, 001, 110, 111\}$, we see that X and Z are not independent for P_A.

Considering possible decompositions of A into set products, we are not so much interested how many continuations into W we have, but if there are any or none. This is often the case in logic, we are not interested how many models there are, but if there is a model at all.

Thus we define independence for A again by:

$\langle X \mid\mid Z\rangle_A$ iff for all $\sigma\tau \in A$ there is $\rho \in A$ such that $\rho(X) = \sigma(X)$ and $\rho(Z) = \tau(Z)$.

The equivalence between probabilitistic independence, $\langle X \mid\mid Z\rangle_{P_A}$ and set independence, $\langle X \mid\mid Z\rangle_A$ is lost now, as the second part of the following example shows:

Example 3.5

(1)

$A := \{000, 010, 100, 110\}$ satisfies both forms of independence, $\langle X \mid\mid Z\rangle_{P_A}$ and set independence, $\langle X \mid\mid Z\rangle_A$.

(2)

$A := \{000, 001, 010, 100, 110\}$.

Here, we have $P_A(X = 0) = 3/5$, $P_A(X = 1) = 2/5$, $P_A(Z = 0) = 3/5$, $P_A(Z = 1) = 2/5$, but $P_A(X = 0, Z = 0) = 2/5 \neq 3/5 * 3/5$.

Consider now $\langle X \mid\mid Z\rangle_A$: Take $\sigma, \tau \in A$, then for all possible values $\sigma(X)$, $\tau(Z)$, there is ρ such that $\rho(X) = \sigma(X)$, $\rho(Z) = \tau(Z)$ - the value $\rho(W)$ is without importance.

We have, however:

Fact 3.2

$\langle X \mid\mid Z\rangle_{P_A} \Rightarrow \langle X \mid\mid Z\rangle_A$.

Proof

Let $\sigma, \tau \in A$, but suppose there is no $\rho \in A$ such that $\rho(X) = \sigma(X)$ and $\rho(Z) = \tau(Z)$. Then $P_A(\sigma(X)), P_A(\tau(Z)) > 0$, but $P_A(\sigma(X), \tau(Z)) = 0$. $\square$

$X \times Y \times Z$ We consider now independence of X and Z, given Y.

The probabilistic definition is:

$\langle X \mid Y \mid Z\rangle_P$ iff for all $x \in X, y \in Y, z \in Z$ $P(x,y,z) * P(y) = P(x,y) * P(y,z)$.

As we are interested mainly in subsets $A \subseteq X \times Y \times Z$ and the resulting P_A, and combination of function fragments, we work immediately with these.

We have to define $\langle X \mid Y \mid Z\rangle_A$.

$\langle X \mid Y \mid Z\rangle_A$ iff for all $\sigma, \tau \in A$ such that $\sigma(Y) = \tau(Y)$ there is $\rho \in A$ such that $\rho(X) = \sigma(X)$, $\rho(Y) = \sigma(Y) = \tau(Y)$, $\rho(Z) = \tau(Z)$.

When we set for $y \in Y$ $A_y := \{\sigma \in A : \sigma(Y) = y\}$, we then have:

$A_y = \{\sigma(X) : \sigma \in A_y\} \times \{y\} \times \{\sigma(Z) : \sigma \in A_y\}$.

The following example shows that $\langle X \mid Y \mid Z\rangle_A$ and $\langle X \mid\mid Z\rangle_A$ are independent from each other:

Example 3.6

(1)

$\langle X \mid Y \mid Z\rangle_A$ may hold, but not $\langle X \mid\mid Z\rangle_A$:

Consider $A := \{000, 111\}$. $\langle X \mid Y \mid Z\rangle_A$ is obvious, as only σ goes through each element in the middle. But there is no 0x1, so $\langle X \mid\mid Z\rangle_A$ fails.

(2)

$\langle X \mid\mid Z\rangle_A$ may hold, but not $\langle X \mid Y \mid Z\rangle_A$:

Consider $A := \{000, 101, 110, 011\}$. Fixing, e.g., 0 in the middle shows that $\langle X \mid Y \mid Z\rangle_A$ fails, but neglecting the middle, we can combine arbitrarily, so $\langle X \mid\mid Z\rangle_A$ holds.

Example 3.7

This example show that $\langle X \mid Y \mid Z\rangle_A$ does not mean that A is some product $A_X \times A_Y \times A_Z$:

Let $A := \{000, 111\}$, then clearly $\langle X \mid Y \mid Z\rangle_A$, but A is no such product.

We have again:

Fact 3.3

Let $\emptyset \neq A \subseteq X \times Y \times Z$, then $\langle X \mid Y \mid Z\rangle_A$ and $\langle X \mid Y \mid Z\rangle_{P_A}$ are equivalent.

Proof

"$\Leftarrow$":

Suppose there are $\sigma, \tau \in A$ such that $\sigma(Y) = \tau(Y)$, but there is no $\rho \in A$ such that $\rho(X) = \sigma(X)$, $\rho(Y) = \sigma(Y) = \tau(Y)$, $\rho(Z) = \tau(Z)$. Then $P_A(\sigma(X), \sigma(Y)), P_A(\tau(Y), \tau(Z)), P_A(\sigma(Y)) > 0$, but $P_A(\sigma(X), \sigma(Y) = \tau(Y), \tau(Z)) = 0$.

"$\Rightarrow$":

Case 1: $P_A(x,y)$ or $P_A(y,z) = 0$, then $P_A(x,y,z) = 0$, and we are done.

Case 2: $P_A(x,y), P_A(y,z) > 0$. By definition and prerequisite, $P_A(x,y) = \frac{card\{\sigma \in A : \sigma(X)=x, \sigma(Y)=y\}}{card(A)} = \frac{card\{\sigma(Z) : \sigma \in A, \sigma(Y)=y\}}{card(A)}$ and $P_A(y,z) = \frac{card\{\sigma \in A : \sigma(Y)=y, \sigma(Z)=Z\}}{card(A)} = \frac{card\{\sigma(X) : \sigma \in A, \sigma(Y)=y\}}{card(A)}$, so $P_A(x,y) * P_A(y,z) = \frac{card\{\sigma \in A : \sigma(Y)=y\}}{card(A) * card(A)}$. Moreover, $P_A(y) = \frac{card\{\sigma \in A : \sigma(Y)=y\}}{card(A)}$, $P_A(x,y,z) = \frac{1}{card(A)}$, so $P_A(y) * P_A(x,y,z) = \frac{card\{\sigma \in A : \sigma(Y)=y\}}{card(A) * card(A)} = P_A(x,y) * P_A(y,z)$

$\square$

$X \times Y \times Z \times W$ The definitions stay the same as for $X \times Y \times Z$.

The equivalence between probabilitistic independence, $\langle X \mid Y \mid Z\rangle_{P_A}$ and set independence, $\langle X \mid Y \mid Z\rangle_A$ is lost again, as the following example shows:

Example 3.8

$A := \{0000, 0001, 0010, 1000, 1010\}$.

Here, we have $P_A(X = 0, Y = 0) = 3/5$, $P_A(X = 1, Y = 0) = 2/5$, $P_A(Y = 0, Z = 0) = 3/5$, $P_A(Y = 0, Z = 1) = 2/5$, $P_A(Y = 0) = 1$, but $P_A(X = 0, Y = 0, Z = 0) = 2/5 \neq 3/5 * 3/5$.

Consider now $\langle X \mid Y \mid Z\rangle_A$: Take $\sigma, \tau \in A$, such that $\sigma(Y) = \tau(Y)$, then for all possible values $\sigma(X)$, $\tau(Z)$, there is ρ such that $\rho(X) = \sigma(X)$, $\rho(Y) = \sigma(Y) = \tau(Y)$, $\rho(Z) = \tau(Z)$ - the value $\rho(W)$ is without importance.

We have, however:

Fact 3.4

$\langle X \mid Y \mid Z\rangle_{P_A} \Rightarrow \langle X \mid Y \mid Z\rangle_A$.

Proof

Let $\sigma, \tau \in A$ such that $\sigma(Y) = \tau(Y)$, but suppose there is no $\rho \in A$ such that $\rho(X) = \sigma(X)$, $\rho(Y) = \sigma(Y) = \tau(Y)$, $\rho(Z) = \tau(Z)$. Then $P_A(\sigma(X), \sigma(Y)), P_A(\sigma(Y), \tau(Z)) > 0$, but $P_A(\sigma(X), \sigma(Y), \tau(Z)) = 0$. $\square$

A remark on generalization The X, Y, Z, W may also be more complicated sets, themselves products, but this will not change definitions and results beyond notation.

In the more complicated cases, we will often denote subsets by more complicated letters than A, e.g., by Σ.

A remark on intuition Consider set independence, where $A := \mu(U)$, $U = U_1 \times \ldots \times U_n$. Set $\langle \ldots \rangle := \langle \ldots \rangle_{\mu(U)}$.

(1) $\langle X \mid\mid Z\rangle$ means then:

- (1.1) all we know is that we are in a normal situation,
- (1.2) if we know in addition something definite about Z (1 model!) we do not know anything more about X, and vice versa.

$\langle X \mid Y \mid Z\rangle$ means then:

- (1.1) all we know is that we are in a normal situation,
- (1.2) if we have definite information about Y, we may know more about X. But knowing something in addition about Z will not give us not more information about X, and conversely.

(2) The restriction to $\mu(U)$ codes our background knowledge.

(3) Note that $X \cup Y \cup Z$ need not be I, e.g., W might be missing. We did not count the continuations into W, but considered only existence of a continuation (if this does not exist, then there just is no such sequence).

This corrsponds to multiplication with 1, the unit ALL on W, or, more generally, in the rest of the paper, with $1_{I-(X \cup Y \cup Z)}$. We may choose however we want, it has to be somewhere, in ALL.

3.1.4 Basic definitions

Definition 3.2

If f is a function, Y a subset of its domain, we write $f \restriction Y$ for the restriction of f to elements of Y.

If F is a set of functions over Y, then $F \restriction Y := \{f \restriction Y : f \in F\}$.

3.2 Probabilistic and set independence

3.2.1 Probabilistic independence

Independence as an abstract ternary relation for probability and other situations has been examined by W. Spohn, see [Spo80], A. P. Dawid, see [Daw79], J. Pearl, see, e.g., [Pea88], etc.

Definition 3.3

(1)

Let $I \neq \emptyset$ be an arbitrary (index) set, for $i \in I$ $U_i \neq \emptyset$ arbitrary sets. Let $U := \Pi\{U_i : i \in I\}$, and for $X \subseteq I$ $U_X := \Pi\{U_i : i \in X\}$.

(2)

Let $P : \mathcal{P}(U) \to [0,1]$ be a probability measure. (We may assume that P is defined by its value on singletons.)

(3.1)

By abuse of language, for $X \subseteq I$, $x \in U_X$, let $P(x) := P(\{u \in U : \forall i \in X u(i) = x(i)\})$, so $P(x) = P(\{u \in U : u \restriction X = x\})$.

Analogously, for $X, Y \subseteq I$, $X \cap Y = \emptyset$, $x \in U_X$, $y \in U_Y$, let $P(x,y) := P(\{u \in U : u \restriction X = x \text{ and } u \restriction Y = y\})$.

(3.2)

Finally, for $X, Y, Z \subseteq I$ pairwise disjoint, $x \in U_X$, $y \in U_Y$, $z \in U_Z$, let $P(x \mid y) := \frac{P(x,y)}{P(y)}$, $P(x \mid y,z) := \frac{P(x,y,z)}{P(y,z)}$, etc.

(We have, of course, to pay attention that we do not divide by 0.)

Definition 3.4

P as above defines a 3-place relation of independence on pairwise disjoint $X, Y, Z \subseteq I$ $\langle X \mid Y \mid Z\rangle_P$ by

$$\langle \mathcal{X} \mid \mathcal{Y} \mid \mathcal{Z}\rangle_P :\leftrightarrow \begin{cases} \forall x \in U_X, \forall y \in U_Y, \forall z \in U_Z (P(y,z) > 0 \to P(x \mid y) = P(x \mid y,z)), & if \quad Y \neq \emptyset \\ i.e., P(x,y)/P(y) = P(x,y,z)/P(y,z), or & \\ P(x,y,z) * P(y) = P(x,y) * P(y,z) & \\ & \\ \forall x \in U_X, \forall z \in U_Z (P(z) > 0 \to P(x) = P(x \mid z)), & if \quad Y = \emptyset \\ i.e., P(x) = P(x,z)/P(z), or & \\ P(x,z) = P(x) * P(z) & \end{cases}$$

If $Y = \emptyset$, we shall also write $\langle X \mid\mid Z\rangle_P$ for $\langle X \mid Y \mid Z\rangle_P$.

Recall from Section 3.1.3 (page 4) that we call this notion probabilistic independence.

E.g., Pearl discusses the rules $(a) - (e)$ of Definition 3.5 (page 9) for the relation defined in Definition 3.4 (page 9).

Definition 3.5

(a) Symmetry: $\langle X \mid Y \mid Z\rangle \leftrightarrow \langle Z \mid Y \mid X\rangle$

(b) Decomposition: $\langle X \mid Y \mid Z \cup W\rangle \to \langle X \mid Y \mid Z\rangle$

(c) Weak Union: $\langle X \mid Y \mid Z \cup W\rangle \rightarrow \langle X \mid Y \cup W \mid Z\rangle$

(d) Contraction: $\langle X \mid Y \mid Z\rangle$ and $\langle X \mid Y \cup Z \mid W\rangle \rightarrow \langle X \mid Y \mid Z \cup W\rangle$

(e) Intersection: $\langle X \mid Y \cup W \mid Z\rangle$ and $\langle X \mid Y \cup Z \mid W\rangle \rightarrow \langle X \mid Y \mid Z \cup W\rangle$

($\emptyset$) Empty outside: $\langle X \mid Y \mid Z\rangle$ if $X = \emptyset$ or $Z = \emptyset$.

Proposition 3.5

If P is a probability measure, and $\langle X \mid Y \mid Z\rangle_P$ defined as above, then $(a) - (d)$ of Definition 3.5 (page 9) hold for $\langle \ldots \rangle = \langle \ldots \rangle_P$, and if P is strictly positive, (e) will also hold.

The proof is elementary, well known, and will not be repeated here.

Doch ein Beispiel geben?

A side remark on preferential structures Being a minimal element is not upward absolute in general preferential structures, but in raked structures, provided the smaller set contains some element minimal in the bigger set.

Fact 3.6

In the probabilistic interpretation, the following holds:

Let U be a finite set, $f : U \rightarrow \Re$ such that $\forall u \in U.f(u) \geq 0$.

For all $A \subseteq U$, such that $\exists a' \in A.f(a') > 0$ and all $a \in A$

$f_A(a) := \frac{f(a)}{\Sigma\{f(a'):a' \in A\}}$ defines a probability measure on A.

For $B \subseteq A$, define $f_A(B) := \Sigma\{f_A(b) : b \in B\}$. Then the following property holds:

(BASIC) For all $D \subseteq B \subseteq A \subseteq U$ such that $\exists b \in B.f(b) > 0$ $f_A(D) = f_A(B) * f_B(D)$.

Proof

For $X \subseteq Y \subseteq U$ such that $\exists y \in Y.f(y) > 0$ we have $f_Y(X) := \Sigma\{f_Y(x) : x \in X\} = \frac{\Sigma\{f(x):x \in X\}}{\Sigma\{f(y):y \in Y\}}$.

Thus, $f_A(D) := \frac{\Sigma\{f(d):d \in D\}}{\Sigma\{f(a):a \in A\}} = \frac{\Sigma\{f(b):b \in B\}}{\Sigma\{f(a):a \in A\}} * \frac{\Sigma\{f(d):d \in D\}}{\Sigma\{f(b):b \in B\}} = f_A(B) * f_B(D)$.

$\square$

We have the following fact for μ generated by a relation:

Fact 3.7

Let U be a finite preferential structure such that for $A \subseteq U$ $\mu(A) = \emptyset \Rightarrow A = \emptyset$.

Then U is ranked iff (BASIC) as defined in Fact 3.6 (page 10) holds for f_A.

Proof

"$\Rightarrow$":

Let $D \subseteq B \subseteq A \subseteq U$, $B \neq \emptyset$.

Case 1: $D \cap \mu(A) = \emptyset$. Then $f_A(D) = 0$.

Case 1.1: If $B \cap \mu(A) = \emptyset$, then $f_A(B) = 0$, and we are done.

Case 1.2: Let $B \cap \mu(A) \neq \emptyset$. If $D \cap \mu(B) = \emptyset$, then $f_B(D) = 0$, and we are done. Suppose $D \cap \mu(B) \neq \emptyset$, so there is $d \in D \cap \mu(B)$, so $d \in D \cap \mu(A)$ by $B \cap \mu(A) \neq \emptyset$ and rankedness, so $f_A(D) \neq \emptyset$, contradiction.

Case 2: $D \cap \mu(A) \neq \emptyset$.

Thus, by $D \subseteq B$, $B \cap \mu(A) \neq \emptyset$, and by rankedness $\mu(B) = B \cap \mu(A)$. So by $D \subseteq B$ again, $D \cap \mu(A) = D \cap (B \cap \mu(A)) = D \cap \mu(B)$. By definition, $f_A(B) := \frac{card(\mu(A) \cap B)}{card(\mu(A))}$, $f_A(D) := \frac{card(\mu(A) \cap D)}{card(\mu(A))}$, $f_B(D) := \frac{card(\mu(B) \cap D)}{card(\mu(B))}$. Thus, $\frac{card(\mu(A) \cap D)}{card(\mu(A))} = \frac{card(\mu(A) \cap B)}{card(\mu(A))} * \frac{card(\mu(B) \cap D)}{card(\mu(B))}$.

"$\Leftarrow$":

Then there are $a, b, c \in U$, where a is incomparable to b, and $b \prec c$ but $a \not\prec c$, or $c \prec b$, but $c \not\prec a$. We have four possible cases.

Let, in all cases, $A := \{a, b, c\}$. We construct a contradiction to (BASIC).

Case 1, $b \prec c$:

Case 1.1, a is incomparable to c : Consider $B := \{a, c\}$, $D := \{a\}$. Then $f_A(D) = \frac{1}{2}$, $f_A(B) = \frac{1}{2}$, $f_B(D) = \frac{1}{2}$.

Case 1.2, $c \prec a$ (so $\prec$ is not transitive): Consider $B := \{a, b\}$, $D := \{a\}$. Then $f_A(D) = 0$, $f_A(B) = 1$, $f_B(D) = \frac{1}{2}$.

Case 2, $c \prec b$:

Case 2.1, a is incomparable to c :

Consider $B := \{a, b\}$, $D := \{a\}$. Then $f_A(D) = \frac{1}{2}$, $f_A(B) = \frac{1}{2}$, $f_B(D) = \frac{1}{2}$.

Case 2.2, $a \prec c$ - similar to Case 1.2.

$\square$

Remark 3.8

Note that sets $A \subseteq B$, where $\mu(B) \cap A = \emptyset$, and sets where $P(A) = 0$ have a similar, exceptional role. This might still be important.

3.2.2 Set independence

We interpret independence here differently, but in a related way, as prepared in Section 3.1.3 (page 4).

Definition 3.6

We consider function sets Σ etc. over a fixed, arbitrary domain $I \neq \emptyset$, into some fixed codomain K.

(1)

For pairwise disjoint subsets X, Y, Z of I, we define

$\langle X \mid Y \mid Z \rangle_\Sigma$ iff for all $f, g \in \Sigma$ such that $f \restriction Y = g \restriction Y$, there is $h \in \Sigma$ such that $h \restriction X = f \restriction X$, $h \restriction Y = f \restriction Y = g \restriction Y$, $h \restriction Z = g \restriction Z$.

Recall from Section 3.1.3 (page 4) that we call this notion set independence.

Y may be empty, then the condition $f \restriction Y = g \restriction Y$ is void.

Note that nothing is said about $I - (X \cup Y \cup Z)$, so we look at the projection of U to $X \cup Y \cup Z$.

When $Y = \emptyset$, we will also write $\langle X \mid\mid Z \rangle_\Sigma$.

$\langle X \mid Y \mid Z \rangle_\Sigma$ means thus, that we can piece functions together, or that we have a sort of decomposition of Σ into a product. This is an independence property, we can put parts together independently.

(2)

In the sequel, we will just write $\langle \ldots \rangle$ for $\langle \ldots \rangle_\Sigma$ when the meaning is clear from the context.

Recall that Example 3.5 (page 6) compares different forms of independence, the probabilistic and the set variant.

Obviously, we can generalize the equivalence results for probabilistic and set independence for $X \times Z$ and $X \times Y \times Z$ to the general situation with W in Section 3.1.3 (page 4), as long as we do not consider the full functions σ, but only their restrictions to X, Y, Z, $\sigma \restriction (X \cup Y \cup Z)$. As we will stop the discussion of probablistic independece here, and restrict ourselves to set independence, this is left as an easy exercise to the reader.

3.3 Basic results for set independence

Notation 3.1

In more complicated cases, we will often write ABC for $\langle A \mid B \mid C\rangle$, and $\neg ABC$ or $-ABC$ if $\langle A \mid B \mid C\rangle$ does not hold. Moreover, we will often just write $f(A)$ for $f \restriction A$, etc.

For $\langle A \cup A' \mid B \mid C\rangle$, we will then write $(AA')BC$, etc.

If only singletons are involved, we will sometimes write abc instead of ABC, etc.

When we speak about fragments of functions, we will often write just $A : \sigma$ for $\sigma \restriction A$, $B : \sigma = \tau$ for $\sigma \restriction B = \tau \restriction B$, etc.

We use the following notations for functions:

Definition 3.7

The constant functions 0_c and 1_c :

$0_c(i) = 0$ for all $i \in I$

$1_c(i) = 1$ for all $i \in I$

Moreover, when we define a function $\sigma : I \to \{0, 1\}$ argument by argument, we abbreviate $\sigma(a) = 0$ by $a = 0$, etc.

Sometimes, we also give (a fragment of) a function just by the sequence of the values, so instead of writing $a = 0$, $b = 1$, $c = 1$, we just write 011 - context will disambiguate.

Remark 3.9

This remark gives an intuitive justification of (some of) above rules in our context.

Rule (a) is trivial.

It is easiest to set $Y := \emptyset$ to see the intuitive meaning.

Rule (b) is a trivial consequence. If we can combine longer sequences, then we can combine shorter, too.

Rule (c) is again a trivial consequence. If we can combine arbitrary sequences, then we can also combine those which agree already on some part.

Rule (d) is the most interesting one, it says when we may combine *longer* sequences. Having just $\langle X \mid\mid Z\rangle$ and $\langle X \mid\mid W\rangle$ as prerequisite does not suffice, as we might lose when applying $\langle X \mid\mid W\rangle$ what we had already by $\langle X \mid\mid Z\rangle$. The condition $\langle X \mid Z \mid W\rangle$ guarantees that we do not lose this.

In our context, it means the following:

We want to combine $\sigma \restriction X$ with $\tau \restriction Z \cup W$. By $\langle X \mid\mid Z\rangle$, we can combine $\sigma \restriction X$ with $\tau \restriction Z$. Fix ρ such that $\rho \restriction X = \sigma \restriction X$, $\rho \restriction Z = \tau \restriction Z$. As $\rho \restriction Z = \tau \restriction Z$, by $\langle X \mid Z \mid W\rangle$, we can combine $\rho \restriction X \cup Z$ with $\tau \restriction W$, and have the result.

Note that we change the functions here, too: we start with σ, τ, then continue with ρ, τ.

We can use what we constructed already as a sort of scaffolding for constructing the rest.

Fact 3.10

Zusammenhang $\langle X \mid Y \mid Z\rangle$ mit Produkten.

Proof

Do

□

We show now that above Rules $(a) - (d)$ hold in our context, but (e) does not hold.

Fact 3.11

In our interpretation,

(1) rule (e) does not hold,

(2) all $\langle X \mid Y \mid \emptyset \rangle$ (and thus also all $\langle \emptyset \mid Y \mid Z \rangle$) hold.

(3) rules $(a) - (d)$ hold, even when one or both of the outside elements of the tripels is the empty set.

Proof

(1) (e) does not hold:

Consider $I := \{x, y, z, w\}$ and $U := \{1111, 0100\}$. Then $x(yw)z$ and $x(yz)w$, as for all $\sigma \restriction yw$ there is just one τ this σ can be. The same holds for $x(yz)w$. But for $y = 1$, there are two different paths through $y = 1$, which cannot be combined.

(2) This is a trivial consequence of the fact that $\{f : f : \emptyset \to U\} = \{\emptyset\}$.

(3) Rules (a), (b), (c) are trivial, by definition, also for $X, Z = \emptyset$. In (c), if $W = \emptyset$, there is nothing to show.

Rule (d): The cases for $X, W, Z = \emptyset$ are trivial. Assume σ, τ such that $\sigma \restriction Y = \tau \restriction Y$, we want to combine $\sigma \restriction X$ with $\tau \restriction Z \cup W$. By $\langle X \mid Y \mid Z \rangle$, there is ρ such that $\rho \restriction X = \sigma \restriction X$, $\rho \restriction Y = \sigma \restriction Y = \tau \restriction Y$, $\alpha \restriction X = \rho \restriction Z = \tau \restriction Z$. Thus ρ and τ satisfy the prerequisite of $\langle X \mid Y \cup Z \mid W \rangle$, and there is α such that $\alpha \restriction X = \rho \restriction X = \sigma \restriction X$, $\alpha \restriction X = \rho \restriction Y = \sigma \restriction Y = \tau \restriction Y$, $\alpha \restriction W = \tau \restriction W$.

$\square$

Next, we give examples which shows that increasing the center set can change validity of the tripel in any way.

Example 3.9

(1)

This example shows that neither $\langle X \mid Y \mid Z \rangle$ implies $\langle X \mid\mid Z \rangle$, nor, conversely, $\langle X \mid\mid Z \rangle$ implies $\langle X \mid Y \mid Z \rangle$.

Consider $I := \{x, y, z\}$.

(1.1) Let $U := \{\langle 0,0,0\rangle, \langle 1,1,1\rangle, \langle 0,1,0\rangle, \langle 1,0,1\rangle, \langle 1,1,0\rangle, \langle 0,0,1\rangle\}$. Then $\langle x \mid\mid z \rangle$, as all combinations for x and y exist, i.e. paths with the projections $\langle 0,0\rangle$, $\langle 0,1\rangle$, $\langle 1,0\rangle$, $\langle 1,1\rangle$. Fix, e.g., $y = 1$. Then the paths through $y = 1$ are $\langle 1,1,1\rangle$, $\langle 0,1,0\rangle$, $\langle 1,1,0\rangle$, but $\langle 0,1,1\rangle$ is missing. So $\langle x \mid y \mid z \rangle$ does not hold.

(1.2) Let $U := \{\langle 0,0,0\rangle, \langle 1,1,1\rangle\}$. Then $\langle x \mid\mid z \rangle$ trivially fails, but $\langle x \mid y \mid z \rangle$ holds.

(2)

Consider $I := \{x, a, b, c, d, z\}$.

Let $\Sigma := \{111111,\ 011110,\ 011101,\ 111100,\ 110111,\ 010000\}$.

Then $\neg x(abcd)z$, $x(abc)z$, $\neg x(ab)z$.

For $\neg x(abcd)z$, fix $abcd = 1111$, then $111111, 011110 \in \Sigma$, but, e.g., $011111 \notin \Sigma$.

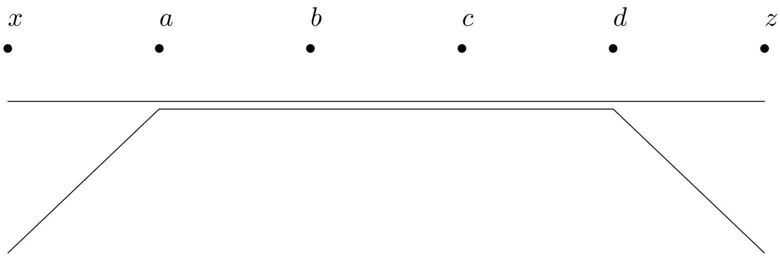
For $x(abc)z$, the following combinations of abc exist: $111, 101, 100$. The result is trivial for 101 and 100. For 111, all combinations for x and z with 0 and 1 exist.

For $\neg x(ab)z$, fix $ab = 10$, then $110111, 010000 \in \Sigma$, but there is, e.g., no $110xy0 \notin \Sigma$.

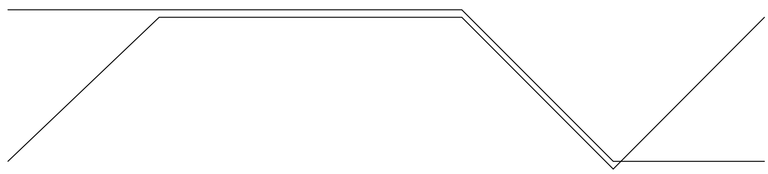
See Diagram 3.1 (page 14)

$\square$

Diagram 3.1

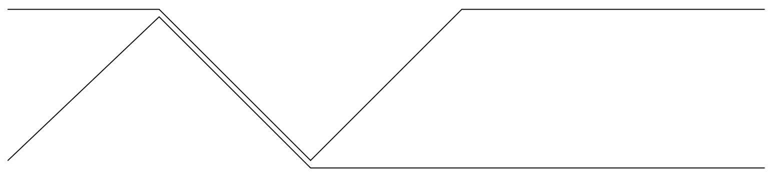


(1) $\neg\langle x \mid abcd \mid z\rangle$



(2) $\langle x \mid abc \mid z\rangle$

add paths equal on abc, different on d, to compensate lacking paths in (1)



(3) $\neg\langle x \mid ab \mid z\rangle$

add paths different on ab, singletons on c, so they don't disturb on abc: seen on abc, the added paths are singletons, so they respect automatically $\langle x \mid abc \mid z\rangle$

3.3.1 Example of a rule derived from the basic rules

We will use the following definition.

Definition 3.8

Given Σ as above, set

$\Sigma_\mu := \{\langle X, Y, Z\rangle : X, Y, Z$ are pairwise disjoint subsets of I, $\langle X \mid Y \mid Z\rangle \notin \Sigma$, but for all $X' \subset X$ and all $Z' \subset Z$ $\langle X' \mid Y \mid Z\rangle \in \Sigma$ and $\langle X \mid Y \mid Z'\rangle \in \Sigma\}$.

We will sometimes write $\langle X, X' \mid Y \mid Z\rangle$ etc. for $\langle X \cup X' \mid Y \mid Z\rangle$.

When we write $\langle X, X' \mid Y \mid Z\rangle$ etc., we will tacitly assume that all sets X, X', Y, Z are pairwise disjoint.

Remark 3.12

(1) Σ_μ contain thus the minimal X and Z for fixed Y, such that $\langle X \mid Y \mid Z\rangle \notin \Sigma$.

(2) By rule (b), for all $\langle X \mid Y \mid Z\rangle \in \Sigma$, there is $\langle X', Y, Z'\rangle \in \Sigma_\mu$ $X \subseteq X'$, $Z \subseteq Z'$, unless all σ, τ such that $\sigma \restriction Y = \tau \restriction Y$ can be combined.

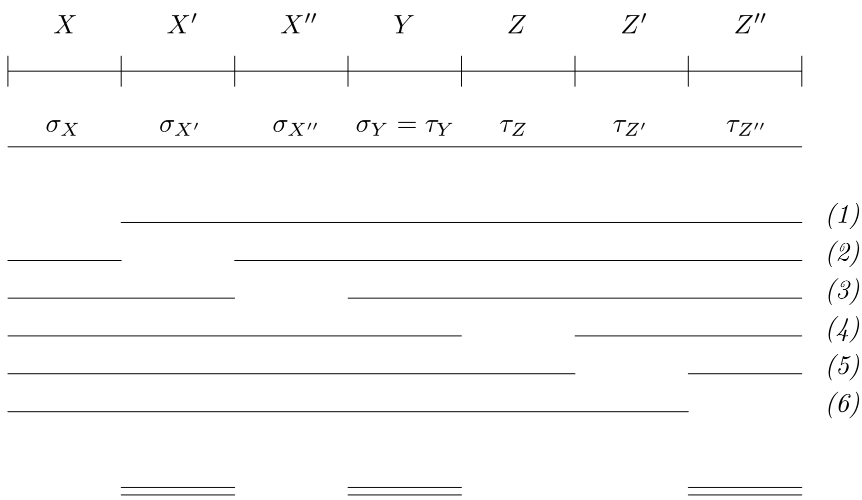
As the cases can become a bit complicated, it is important to develop a good intuition and representation of the problem. We do this now in the proof of the following fact, where we use the result we want to prove to guide our intuition.

Fact 3.13

Let Σ be closed under rules $(a) - (d)$. Then, if $\langle X, X', X'' \mid Y \mid Z, Z', Z''\rangle \in \Sigma_\mu$, then $\langle X, Z' \mid X', Y, Z'' \mid X'', Z\rangle \notin \Sigma$.

Proof

Diagram 3.2



Prerequisite: $\sigma_{X'} = \tau_{X'}$, $\sigma_Y = \tau_Y$, $\sigma_{Z''} = \tau_{Z''}$

The upper line is the final aim. Line (1) expresses that we can combine all parts except s_X, by $\langle X', X'' \mid Y \mid Z, Z', Z'' \rangle$, which holds by $\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma_\mu$, by similar arguments, we can combine as indicated in lines (2) – (6). We now assume $\langle X, Z' \mid X', Y, Z'' \mid X'', Z \rangle \in \Sigma$. So we have to look at fragments, which agree on X', Y, Z''. This is, for instance, true for (1) and (3).

We turn this argument now into a formal proof:

Assume

(A) $\langle X, Z' \mid X', Y, Z'' \mid X'', Z \rangle \in \Sigma$, and

(B) $\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma_\mu$.

(C) $\langle X, X' \mid Y \mid Z, Z', Z'' \rangle$ by (B), see line (3)

(D) $\langle X \mid X', Y, Z', Z'' \mid X'', Z \rangle$ by (A) and rule (c)

(E) $\langle X \mid X', Y \mid Z, Z', Z'' \rangle$ by (C) and rule (c)

(F) $\langle X \mid X', Y \mid Z', Z'' \rangle$ by (E) and (b)

(G) $\langle X \mid X', Y \mid X'', Z, Z', Z'' \rangle$ by (D) and (F) and (d)

(K) $\langle X \mid X', X'', Y \mid Z, Z', Z'' \rangle$ by (G) and (c)

(L) $\langle X', X'' \mid Y \mid Z, Z', Z'' \rangle$ by (B), see line (1)

(M) $\langle Z, Z', Z'' \mid X', X'', Y \mid X \rangle$ by (K) and (a)

(N) $\langle Z, Z', Z'' \mid Y \mid X', X'' \rangle$ by (L) and (a)

(O) $\langle Z, Z', Z'' \mid Y \mid X, X', X'' \rangle$ by (M) and (N) and (d)

(P) $\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle$ by (O) and (a).

So we conclude $\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma$, a contradiction.

Comment:

We first move Z', Z'' to the right, and then X', X'' to the left.

Moving Z', Z'' :

We use X'' (or Z) on the right, which not be changed, therefore we can use line (3), resulting in

(C) $\langle X, X' \mid Y \mid Z, Z', Z'' \rangle$, or, directly

(C') $\langle X, X' \mid Y \mid Z', Z'' \rangle$, again by Σ_μ,

which is modified to

(F) $\langle X \mid X', Y \mid Z', Z'' \rangle$, so we have on the right Z', Z'' which we want to move.

We put Z' in the middle (Z'' is there already) of (A), resulting in

(D) $\langle X \mid X', Y, Z', Z'' \mid X'', Z \rangle$.

Now we can apply (d) to (D) and (F), and have moved Z', Z'' to the right:

(G) $\langle X \mid X', Y \mid X'', Z, Z', Z'' \rangle$.

We still have to move X' and X'' to the left of (G), and do this in an analogous way.

□

Note that our results stays valid, if some of the X', X'', Z', Z'' are empty.

Aber resultat darf nicht links oder rechts $\emptyset$ sein.

Corollary 3.14

Let Σ be closed under rules $(a) - (d)$. Then, if $\langle X, X', X'' \mid Y, Y', Y'' \mid Z, Z', Z'' \rangle \in \Sigma_\mu$, then $\langle X, Y', Z' \mid X', Y, Z'' \mid X'', Y'', Z \rangle \notin \Sigma$.

Thus, if, for given $Y \cup Y' \cup Y''$, $\langle X, X', X'' \mid Y, Y', Y'' \mid Z, Z', Z'' \rangle \in \Sigma_\mu$, then for no distribution of $X \cup X' \cup X'' \cup$

$Y \cup Y' \cup Y'' \cup Z \cup Z' \cup Z''$ such that the outward elements are non-empty, $\langle X, Y', Z' \mid X', Y, Z'' \mid X'', Y'', Z \rangle \in \Sigma$.

Proof

Suppose $\langle X, Y', Z' \mid X', Y, Z'' \mid X'', Y'', Z \rangle \in \Sigma$. Then by rule (c) $\langle X, Z' \mid X', Y, Y', Y'', Z'' \mid X'', Z \rangle \in \Sigma$. Set $Y_1 := Y \cup Y' \cup Y''$. Then $\langle X, Z' \mid X', Y_1, Z'' \mid X'', Z \rangle \in \Sigma$, and $\langle X, X', X'' \mid Y_1 \mid Z, Z', Z'' \rangle \in \Sigma_\mu$, contradicting Fact 3.13 (page 15). $\square$

Validity of $ABC, ACD, ADE, AEB \Rightarrow ABE$						
	A	B	C	D	E	
	σ	$\sigma = \tau$			τ	$ABE?$
(1) ρ_1	σ	$\sigma = \tau$	τ			ABC
(2) ρ_2	σ		$\rho_1 = \tau$	τ		ACD
(3) ρ_3	σ			$\rho_2 = \tau$	τ	ADE
(4) ρ_4	σ	$\sigma = \tau$			$\rho_3 = \tau$	AEB

3.4 Examples of new rules

3.4.1 New rules

Above rules $(a) - (d)$ are not the only ones to hold, and we introduce now more complicated ones, and show that they hold in our situation. Of the possibly infinitary rules, only (Loop1) is given in full generality, (Loop2) is only given to illustrate that even the infinitary rule (Loop1) is not all there is.

For warming up, we consider the following short version of (Loop1):

Example 3.10

$ABC, ACD, ADE, AEB \Rightarrow ABE.$

We show that this rule holds in all Σ.

Suppose $A : \sigma$, $B : \sigma = \tau$, $C : \tau$, so by ABC, there is ρ_1 such that

$A : \rho_1 = \sigma$, $B : \rho_1 = \sigma = \tau$, $C : \rho_1 = \tau$. So by ACD, there is ρ_2 such that

$A : \rho_2 = \sigma$, $C : \rho_2 = \rho_1 = \tau$, $D : \rho_2 = \tau$. So by ADE, there is ρ_3 such that

$A : \rho_3 = \sigma$, $D : \rho_3 = \rho_2 = \tau$, $E : \rho_3 = \tau$. So by AEB, there is ρ_4 such that

$A : \rho_4 = \sigma$, $E : \rho_4 = \rho_3 = \tau$, $B : \rho_4 = \tau = \sigma$.

So ABE.

We abbreviate this reasoning by:

(1) $ABC : A : \sigma$, $B : \sigma = \tau$, $C : \tau$

(2) $ACD : (1) + \tau$

(3) $ADE : (2) + \tau$

(4) $AEB : (3) + \tau$

So ABE.

It is helpful to draw a little diagram as in the following Table 3.4.1 (page 18).

.

We introduce now some new rules.

Definition 3.9

- (Bin1)
  $XYZ, XY'Z, Y(XZ)Y' \Rightarrow X(YY')Z$
- (Bin2)
  $XYZ, XZY', Y(XZ)Y' \Rightarrow X(YY')Z$
- (Loop1)
  $AB_1B_2, \ldots, AB_{i-1}B_i, AB_iB_{i+1}, AB_{i+1}B_{i+2}, \ldots, AB_{n-1}B_n, AB_nB_1 \Rightarrow AB_1B_n$

so we turn AB_nB_1 around to AB_1B_n.

When we have to be more precise, we will denote this condition $(Loop1_n)$ to fix the length.

- (Loop2)

  $ABC, ACD, DAE, DEF, FDG, FGH, HFB \Rightarrow HBF$:

The complicated structure of these rules suggests already that the ternary relations are not the right level of abstraction to speak about construction of functions from fragments. This is made formal by our main result below, which shows that there is no finite characterization by such relations. In other words, the main things happen behind the screen.

Fact 3.15

The new rules are valid in our situation.

Proof

- (Bin1)

  (1) $XYZ : X : \sigma,\ Y : \sigma = \tau,\ Z : \tau$

  (2) $XY'Z : X : \sigma,\ Y' : \sigma = \tau,\ Z : \tau$

  (3) $Y(XZ)Y' : (1) + (2)$

  So $X(YY')Z$.

- (Bin2)

  Let $X : \sigma,\ Y : \sigma = \tau,\ Y' : \sigma = \tau,\ Z : \tau$

  (1) $XYZ : X : \sigma,\ Y : \sigma = \tau,\ Z : \tau$

  (2) $XZY' : (1) + \tau$

  (3) $Y(XZ)Y' : (1) + (2)$

  So $X(YY')Z$.

- (Loop1)

  (1) $AB_1B_2 : A : \sigma,\ B_1 : \sigma = \tau,\ B_2 : \tau$

  (2) $AB_2B_3 : (1) + \tau$

  ....

  (i-1) $AB_{i-1}B_i : (i-2) + \tau$

  (i) $AB_iB_{i+1} : (i-1) + \tau$

  $(i+1)$ $AB_{i+1}B_{i+2} : (i) + \tau$

  ....

  $(n-1)$ $AB_{n-1}B_n : (n-2) + \tau$

  (n) $AB_nB_1 : (n-1) + \tau$

  So AB_1B_n.

- (Loop2)

  Let

  (1) $ABC : A : \sigma,\ B : \sigma = \tau,\ C : \tau$

  (2) $ACD : 1 + \tau$

  (3) $DAE : 2 + \sigma$

  (4) $DEF : 3 + \sigma$

(5) $FDG : 4 + \tau$

(6) $FGH : 5 + \tau$

(7) $HFB : 6 + \sigma$

So HBF by $B : \sigma = \tau$.

Note that we use here $B : \sigma = \tau$, $E : \sigma = \tau$, $H : \sigma = \tau$, whereas the other tripels are used for other functions.
□

Next we show that the full (Loop1) cannot be derived from the basic rules $(a) - (d)$ and (Bin1), and shorter versions of (Loop1). (This is also a consequence of the sequel, but we want to point it out right away.)

Fact 3.16

Let $n \geq 1$, then $(Loop1_n)$ does not follow from the rules $(a) - (d)$, $(\emptyset)$, (Bin1), and the shorter versions of (Loop1)

Proof

Consider the following set of tripels $L \cup L'$ over $I := \{a, b_1, \ldots, b_n\}$:

$L := \{ab_1b_2, \, \ldots, \, ab_ib_{i+1}, \, \ldots, \, ab_{n-1}b_n, \, ab_nb_1\}$,

$L' := \{\emptyset AB : A \cap B = \emptyset, \, A \cup B \subseteq I\}$,

and close this set under symmetry (rule (a)). Call the resulting set $\mathcal{A}$.

Note that, on the outside, we have $\emptyset$ or singletons, inside singletons or $\emptyset$. If the inside is $\emptyset$, one of the outside sets must also be $\emptyset$.

When we look at L, and define a relation $<$ by $x < y$ iff $axy \in L$, we see that the only $<$-loop is $b_1 < b_2 < \ldots < b_n < b_1$.

We show first that $\mathcal{A}$ is closed under rules $(a) - (d)$ (see Definition 3.5 (page 9)).

(a) is trivial.

(b) If $W = \emptyset$ or $Z = \emptyset$, this is trivial, if $W = Z$, this is trivial, too.

(c) If $Z \cup W = \emptyset$, this is trivial, if $Z \cup W$ is a singleton, so $Z = \emptyset$ or $W = \emptyset$ or $Z = W$. $Z = \emptyset$ or $W = \emptyset$ are trivial, otherwise $Z = W$ contradicts disjointness.

(d) $Z = \emptyset$ is trivial, so is $W = \emptyset$, otherwise $Z = W$ contradicts disjointness.

(Bin1) $X = \emptyset$ or $Z = \emptyset$ are trivial, otherwise $X = Z$ is excluded by disjointness. So we are in L' for $Y(XZ)Y'$. So $Y = \emptyset$ or $Y' = \emptyset$ and it is trivial.

Obviously, $(Loop1_n)$ does not hold.

We show now that all $(Loop1_k)$, $0 \leq k < n$ hold.

The cases $n = 1$, $n = 2$ are trivial.

Consider the case $2 < k < n$.

This has the form $AB_1B_2, AB_2B_3, \ldots, AB_{k-1}B_k, AB_kB_1 \Rightarrow AB_1B_k$.

If $A = \emptyset$ or $B_k = \emptyset$, the condition holds.

So assume $A, B_k \neq \emptyset$. Thus, by above remark, descending to B_{k-1} etc., we see that all $B_i \neq \emptyset$, $1 \leq i \leq k$. Thus, all prerequisites are in L. Moreover, A has to be a, which is the only element occuring repeatedly on the outside. Consider now the relation $<'$ defined by $U <' V$ iff AUV is among the prerequisites. We then have $B_1 <' B_2 <' \ldots <' B_k <' B_1$, where all B_i are some b_j, we see that the resulting $<'$-loop is too short, so the prerequisites cannot hold, and we have a contradiction.

□

3.5 There is no finite characterization

We turn to our main result.

3.5.1 Discussion

Consider the following simple, short, loop for illustration:

$ABC, ACD, ADE, AEF, AFG, AGB \Rightarrow ABG$ - so we can turn AGB around to ABG.

Of course, this construction may be arbitrarily long.

The idea is now to make ABG false, and, to make it coherent, to make one of the interior conditions false, too, say ADE. We describe this situation fully, i.e. enumerate all conditions which hold in such a situation. If we make now ADE true again, we know this is not valid, so any (finite) characterization must say "NO" to this. But as it is finite, it cannot describe all the interior tripels of the type ADE in a sufficiently long loop, so we just change one of them which it does not "see" to FALSE, and it must give the same answer NO, so this fails.

Basically, we cannot describe parts of the loop, as the <||>-language is not rich enough to express it, we see only the final outcome.

The problem is to fully describe the situation.

3.5.2 Composition of layers

A very helpful fact is the following:

Definition 3.10

Let Σ_j be function sets over I into some set K, $j \in J$.

Let $\Sigma := \{ \ f : I \rightarrow K^J : f(i) = \{\langle f_j(i), j\rangle : j \in J, f_j \in \Sigma_j\} \ \}$.

So any $f \in \Sigma$ has the form $f(i) = \langle f_1(i), f_2(i), \ldots, f_n(i)\rangle$, $f_m \in \Sigma_m$ (we may assume J to be finite).

Thus, given $f \in \Sigma$, $f_m \in \Sigma_m$ is defined.

Fact 3.17

For the above Σ $\langle A \mid B \mid C\rangle$ holds iff it holds for all Σ_j.

Thus, we can destroy the $\langle A \mid B \mid C\rangle$ independently, and collect the results.

Proof

The proof is trivial, and a direct consequence of the fact that $f = f'$ iff for all components $f_j = f'_j$.

Suppose for some Σ_k, $k \in J$, $\neg\langle A \mid B \mid C\rangle$.

So for this k there are $f_k, f'_k \in \Sigma_k$ such that $f_k(B) = f'_k(B)$, but there is no $f''_k \in \Sigma_k$ such that $f''_k(A) = f_k(A)$, $f''_k(B) = f_k(B) = f'_k(B)$, $f''_k(C) = f'_k(C)$ (or conversely). Consider now some $h \in \Sigma$ such that $h_k = f_k$, and h' is like h, but $h'_k = f'_k$, so also $h' \in \Sigma$. Then $h(B) = h'(B)$, but there is no $h'' \in \Sigma$ such that $h''(A) = h(A)$, $h''(B) = h(B) = h'(B)$, $h''(C) = h'(C)$.

Conversely, suppose $\langle A \mid B \mid C\rangle$ for all Σ_j. Let $h, h' \in \Sigma$ such that $h(B) = h'(B)$, so for all $j \in J$ $h_j(B) = h'_j(B)$, where $h_j \in \Sigma_j$, $h'_j \in \Sigma_j$, so there are $h''_j \in \Sigma_j$ with $h''_j(A) = h_j(A)$, $h''_j(B) = h_j(B) = h'_j(B)$, $h''_j(C) = h'_j(C)$ for all $j \in J$. Thus, h'' composed of the h''_j is in Σ, and $h''(A) = h(A)$, $h''(B) = h(B) = h'(B)$, $h''(C) = h'(C)$.

$\square$

3.5.3 Systematic construction

Recall the general form of (Loop1) for singletons:

$ab_1b_2, \ldots, ab_{i-1}b_i, ab_ib_{i+1}, ab_{i+1}b_{i+2}, \ldots, ab_{n-1}b_n, ab_nb_1 \Rightarrow ab_1b_n$

We will fully describe a model of above tripels, with the exception of ab_1b_n and ab_ib_{i+1} which will be made to fail, and all other $\langle X \mid Y \mid Z\rangle$ which are not in above list of tripels to preserve, will fail, too (except for $X = \emptyset$ or $Z = \emptyset$).

Thus, the tripels to preserve are:

$P := \{ab_1b_2, \ldots, ab_{i-1}b_i,$ (BUT NOT ab_ib_{i+1}) $, ab_{i+1}b_{i+2}, \ldots, ab_{n-1}b_n, ab_nb_1\}$

We use the following fact:

Fact 3.18

Let $X \subseteq I$, $card(X) > 1$, $\Sigma_X := \{\ \sigma : I \to \{0,1\} : card\{x \in X : \sigma(x) = 0\}$ is even $\}$

Then $\neg ABC$ iff $A \cap X \neq \emptyset$, $C \cap X \neq \emptyset$, $X \subseteq A \cup B \cup C$.

Proof

"$\Leftarrow$":

Suppose $A \cap X \neq \emptyset$, $C \cap X \neq \emptyset$, $X \subseteq A \cup B \cup C$.

Take σ such that $card\{x \in X : \sigma(x) = 0\}$ is odd, then $\sigma \notin \Sigma_X$. As $X \not\subseteq A \cup B$, there is $\tau \in \Sigma_X$ such that $\sigma \restriction A \cup B = \tau \restriction A \cup B$. As $X \not\subseteq B \cup C$, there is $\rho \in \Sigma_X$ such that $\rho \restriction B \cup C = \sigma \restriction B \cup C$. Thus, $\tau \restriction B = \rho \restriction B$. If there were $\alpha \in \Sigma_X$ such that $\alpha \restriction A \cup B = \tau \restriction A \cup B$ and $\alpha \restriction B \cup C = \rho \restriction B \cup C$, then $\alpha \restriction A \cup B \cup C = \sigma \restriction A \cup B \cup C$, contradiction

"$\Rightarrow$":

Suppose $A \cap X = \emptyset$ or $C \cap X = \emptyset$, or $X \not\subseteq A \cup B \cup C$. We show ABC.

Case 1: $C \cap X = \emptyset$. Let $\sigma, \tau \in \Sigma_X$ such that $\sigma \restriction B = \tau \restriction B$. As $C \cap X = \emptyset$, we can continue $\sigma \restriction A \cup B$ as we like.

Case 2, $A \cap X = \emptyset$, analogous.

Case 3: $X \not\subseteq A \cup B \cup C$. But then there is no restriction in $A \cup B \cup C$.

□

We will have to make ab_1b_n false, but ab_nb_1 true. On the other hand, we will make ab_1b_3 false, but ab_3b_1 need not be preserved.

This leads to the following definition, which helps to put order into the cases.

Definition 3.11

Suppose we have to destroy axy. Then

$dmin(axy) := min\{d(\{a, x, y\}, \{a, u, v\}) : auv$ has to be preserved $\}$ - d the counting Hamming distance.

Thus, $dmin(ab_1b_n) = 0$ (as ab_nb_1 has to be preserved), $dmin(ab_1b_3) = 1$ (because ab_1b_2 has to be preserved, but not ab_3b_1).

We introduce the following order defined from the loop prerequisites to be preserved.

Definition 3.12

Order the elements by following the string of sequences to be preserved as follows:

$b_{i+1} \prec b_{i+2} \prec \ldots \prec b_{n-1} \prec b_n \prec b_1 \prec b_2 \prec \ldots \prec b_{i-1} \prec b_i$

Note that the interruption at ab_ib_{i+1} is crucial here - otherwise, there would be a cycle.

As usual, $\preceq$ will stand for $\prec$ or $=$.

3.5.4 The cases to consider

The elements to consider are: $a, b_1, \ldots, b_n$.

Recall that the tripels to preserve are:

$P := \{ab_1b_2, \ldots, ab_{i-1}b_i$, (BUT NOT ab_ib_{i+1}) $, ab_{i+1}b_{i+2}, \ldots, ab_{n-1}b_n, ab_nb_1\}$

The $\langle X \mid Y \mid Z\rangle$ to destroy are (except when $X = \emptyset$ or $Z = \emptyset$) :

(1) all $\langle X \mid\mid Z\rangle$

(2) all $\langle X \mid Y \mid Z\rangle$ such that $X \cup Y \cup Z$ has > 3 elements

(3) all tripels which do not have a on the outside, e.g. bgc

(4) and the following tripels:

(the (0) will be explained below - for the moment, just ignore it)

$ab_1b_3, \ldots, ab_1b_{n-1}$, ab_1b_n (0)

ab_2b_1 (0), $ab_2b_4, \ldots, ab_2b_n$

ab_3b_1, ab_3b_2 (0), $ab_3b_5, \ldots, ab_3b_n$

....

ab_ib_1, $ab_ib_2, , \ldots,$ ALSO $ab_ib_{i+1}, \ldots, ab_ib_n$

....

$ab_{n-2}b_1, , \ldots,$ $ab_{n-2}b_{n-3}$ (0), $ab_{n-2}b_n$

$ab_{n-1}b_1, , \ldots,$ $ab_{n-1}b_{n-2}$ (0),

ab_nb_1, $, \ldots, ab_nb_{n-1}$ (0)

3.5.5 Solution of the cases

We show how to destroy all tripels mentioned above, while preserving all tripels in P.

(1) all $\langle X \mid Y \mid Z\rangle$ where $X \cup Y \cup Z$ has > 3 elements:

See Fact 3.18 (page 22) with the X there with 4 elements, for all such X, Y, Z separately, so all tripels in P are preserved.

(2) all $\langle X \mid Y \mid Z\rangle$ with 1 element: -

(3) all $\langle X \mid\mid Z\rangle$:

This can be done by considering $\Sigma_j := \{0_c, 1_c\}$. Then, say for a, c, we have to examine the fragments 00 and 11, but there is no 10 or 01. For $\langle a \mid b \mid c\rangle$ this is no problem, as we have only the two 000, 111, which do not agree on b.

(4) all $\langle X \mid Y \mid Z\rangle$ with 2 elements: eliminated by $\langle X \mid\mid Z\rangle$

(5) all $\langle X \mid Y \mid Z\rangle$ with 3 elements:

(5.1) a is not on the outside

(5.1.1) a is in the middle, we need $\neg xay$: Consider Σ with 2 functions, 0_c, and the second defined by $a = 0$, and all $u = 1$ for $u \neq a$. Obviously, $\neg xay$. Recall that all tripels to be preserved have a on the outside, and some other element x in the middle. Then the two functions are different on x.

(5.1.2) a is not in xyz, we need $\neg xyz$: Consider Σ with 2 functions, 0_c, and the second defined by $a = y = 0$, all $u = 1$ for $u \neq a$, $u \neq y$. As a is neither x nor z, $\neg xyz$. If some uvw has a on the outside, say $u = a$, then both functions are 000 or 0vw on this tripel, so uvw holds.

(5.2) a is on the outside, we destroy ayz :

(5.2.1) Case $dmin(ayz) > 0$:
Take as Σ the set of all functions with values in $\{0,1\}$, but eliminate those with $a = y = z = 0$. Then $\neg ayz$ (we have $100, 001, 101$, but not 000), but for all auv with $d(\{a,y,z\},\{a,u,v\}) > 0$ auv has all possible combinations, as all combinations for ay and az exist.

(5.2.2) Case $dmin(ayz) = 0$.
The elements with $dmin = 0$ are:
ab_1b_n, ab_2b_1, $\ldots$, ab_ib_{i-1}, NOT $ab_{i+1}b_i$, $ab_{i+2}b_{i+1}$, $\ldots$, $ab_{n-1}b_{n-2}$, ab_nb_{n-1}, they were marked with (0) above.
Σ will again have 2 functions, the first is always 0_c.
The second function: Always set $a = 1$.
We see that the tripels with $dmin = 0$ to be destroyed have the form ayz, where z is the immediate $\prec$-predecessor of y in above order - see Definition 3.12 (page 22). Conversely, those to be preserved (in P) have the form azy, where again z is the immediate $\prec$-predecessor of y.
We set $z' = 1$ for all $z' \preceq z$, and $y' = 0$ for all $y' \succeq y$. Recall that $z \prec y$, so we have the picture $b_{i+1} = 1, \ldots, z = 1, y = 0, \ldots, b_i = 0$.
Then $\neg ayz$, as we have the fragments 000, 101. But azy, as we have the fragments 000, 110. Moreover, considering the successors of the sequence, we give the values 11, or 10, or 00. This results in the function fragments for auv as 111, or 110, or 100. But the resulting fragment sets (together with 0_c) are then: $\{000, 111\}$, $\{000, 110\}$, $\{000, 100\}$. They all make auv true. Thus, all tripels in P are preserved.

3.6 Systematic construction of new rules

This section is an outline - not a formal proof - for constructing a complete rule set for our scenario.

We give here a general way how to construct new rules of the type ABC, DEF, ... $\Rightarrow$ XYZ which are valid in our situation.

3.6.1 Consequences of a single tripel

Let $(XX'X'')Y(ZZ'Z'')$ be a tripel, then all consequences of this single tripel have the form $X(X'YZ')Z$ (up to symmetry).

Obviously, such $X(X'YZ')Z$ are consequences, using rules (b) and (c).

We now give counterexamples to other forms, to show that they are not consequences in our setting. We always assume that the outside is not $\emptyset$. We consider $A = B = C = \{0, 1\}$, and subsets of $A \times B \times C$.

(1) Y decreases:
Consider $\{000, 111\}$, then ABC, but not $A\emptyset C$.

(2) Z increases:
Consider $\{000, 101\}$, then $A\emptyset B$, but not $A\emptyset(BC)$.

(3) X goes from left to right:
Consider $\{000, 110\}$, then (AB)C, but not $A(BC)$

(4) Y increases by some arbitrary W :
Consider $\{000, 101, 110, 011\}$, then $A\emptyset C$, but not ABC.

3.6.2 Construction of function trees

We can construct new functions from two old functions using tripels ABC, so, in a more general way, we have a binary function construction tree, where the old functions are the leaves, and the new function is the root. The form of such a tree is obvious, the tripels used are either directly given, or consequences of such tripels. In Example 3.13 (page 29), for instance, in the construction of ρ_2, we used ACD, but we could also have used e.g. $AC(DD')$, for some D'.

3.6.3 Derivation trees

Not all such function construction trees are proof trees for a rule $T_1, \ldots, T_n \Rightarrow T$, where the T_i and T are tripels.

We have to look at the logical structure of the tripels to see what we need. In order to show $T = ABC$, we assume given two arbitrary functions σ and τ, which agree on B, and construct ρ such that on A $\rho = \sigma$, on B $\rho = \sigma = \tau$ (the latter, $\sigma = \tau$ by prerequisite), and on C $\rho = \tau$. We will write this as $A : \rho = \sigma$, $B : \rho = \sigma = \tau$, $C : \rho = \tau$.

Thus, we have no functions at the beginning, except σ and τ, so all leaves in a proof tree for $T_1, \ldots, T_n \Rightarrow T$ have to be σ or τ. Moreoever, all we know about σ and τ is that they agree on B. Thus, we can only use some $T_i' = A'B'C'$ on σ and τ if $B' \subseteq B$. Likewise, in the interior of the tree, we can only use $\sigma \upharpoonright B = \tau \upharpoonright B$, and, of course, all equalities which hold be construction. E.g., in Example 3.13 (page 29), in the construction of ρ_2, by construction of ρ_1, $C : \rho_1 = \tau$, so we can use ACD to construct ρ_2 from ρ_1 and τ.

At the root, we must have a function ρ of the form $A : \rho = \sigma$, $B : \rho = \sigma = \tau$, $C : \rho = \tau$. In Example 3.13 (page 29), ρ_4, at the root, was constructed using AEB from ρ_3 and τ. But we do not interpret ρ_4 as AEB, but as ABE, which is possible, as $A : \rho_4 = \sigma$, $B : \rho_4 = \sigma = \tau$, $E : \rho_4 = \tau$.

Intermediate nodes can be read as an intermediate result $A'B'C'$ by the same criteria: They must be functions ρ' such that $A' : \rho' = \sigma$, $B' : \rho' = \sigma = \tau$, $C' : \rho' = \tau$ and all B'' such that $B'' : \sigma = \tau$ used up to this node must be subsets of B', as $B' : \sigma = \tau$ is the only hypothesis we then have.

3.6.4 Examples

Diagram 3.3

$$\begin{tikzcd}[column sep=small]
\sigma \arrow[dr, no head] & & \tau \arrow[dl, no head] & \\
& \rho_1 \arrow[dr, no head] & & \tau \arrow[dl, no head] \\
& & \rho_2 &
\end{tikzcd}$$

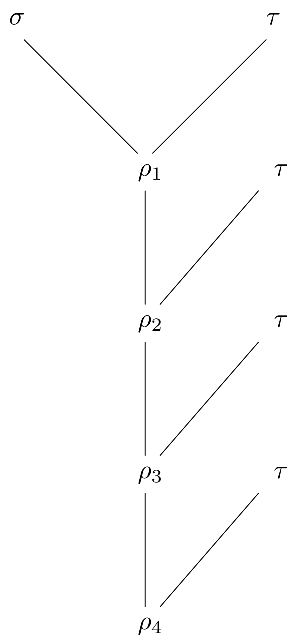
Example 3.11

$$\begin{tikzcd}[column sep=small]
\sigma \arrow[dr, no head] & & \tau \arrow[dl, no head] & & \sigma \arrow[dr, no head] & & \tau \arrow[dl, no head] \\
& \rho_1 \arrow[drr, no head] & & & & \rho_2 \arrow[dll, no head] & \\
& & & \rho_3 & & &
\end{tikzcd}$$

Examples 3.12 and 3.14

Diagram 3.4

Example 3.13



ρ_1, *using* ABC - $A:\sigma, B:\sigma=\tau, C:\tau$

ρ_2, *using* ACD - $A:\rho_1=\sigma, C:\rho_1=\tau, D:\tau$

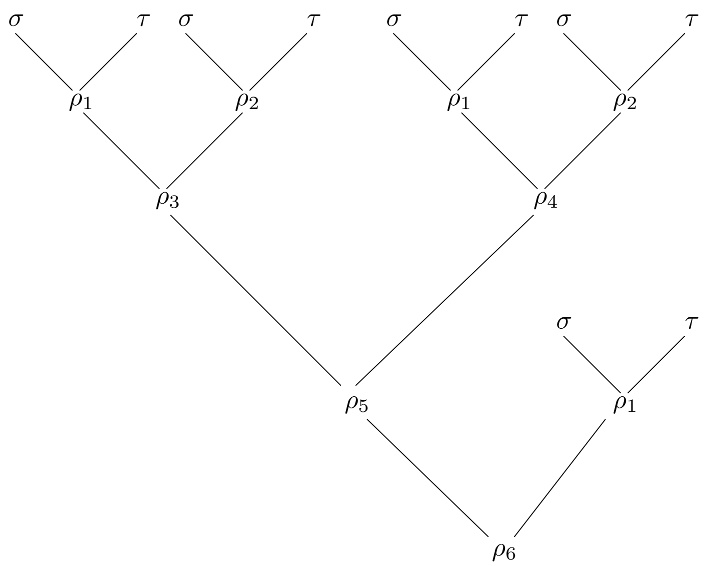
ρ_3, *using* ADE - $A:\rho_2=\sigma, D:\rho_2=\tau, E:\tau$

ρ_4, *using* AEB - $A:\rho_3=\sigma, E:\rho_3=\tau, B:\sigma=\tau$

Interpretation: ABE, *common part* $B:\sigma=\tau$

Diagram 3.5

Example 3.15



Explanation:

By "prerequisite" of ρ_i we mean the set X we used in the construction, where $X : \sigma = \tau$. For instance, in the construction of ρ_2 in Example 3.11 (page 29), we used only that $B \cup C : \rho_1 = \tau$ by the construction of ρ_1, no additional use of some $\sigma = \tau$ was made.

By "common part" of ρ_i we mean the set X such that $X : \rho_i = \sigma = \tau$.

Example 3.11

(Contraction), ABC, $A(BC)D \to AB(CD)$:

(See Diagram 3.3 (page 26) upper part.)

- $\rho_1 : A : \sigma$, $B : \sigma = \tau$, $C : \tau$

  generated by ABC from σ, τ

  prerequisite B,

  common part: B

  ρ_1 can be interpreted as the (trivial) derived tripel ABC

- $\rho_2 : A : \rho_1 = \sigma$, $B : \rho_1 = \sigma = \tau$, $C : \rho_1 = \tau$, $D : \tau$

  generated by $A(BC)D$ from ρ_1, τ

  prerequisite -,

  common part: B.

  ρ_2 can be interpreted as a derived tripel by $AB(CD)$.

  ρ_2 can also be interpreted as a derived tripel by $A(BC)D$ or $A(BD)C$. Note that these possibilities can be derived from $AB(CD)$ by rule (c), Weak Union.

Example 3.12

(Bin1), XYZ, $XY'Z$, $Y(XZ)Y' \Rightarrow X(YY')Z$:

(See Diagram 3.3 (page 26) lower part.)

- $\rho_1 : X : \sigma$, $Y : \sigma = \tau$, $Z : \tau$

  generated by XYZ from σ, τ

  prerequisite Y

  common part: Y

- $\rho_2 : X : \sigma$, $Y' : \sigma = \tau$, $Z : \tau$

  generated by $XY'Z$ from σ, τ

  prerequisite Y'

  common part: Y'

- $\rho_3 : Y : \rho_1 = \sigma = \tau$, $X : \rho_1 = \rho_2 = \sigma$, $Z : \rho_1 = \rho_2 = \tau$, $Y' : \rho_2 = \sigma = \tau$

  generated by $Y(XZ)Y'$ from ρ_1, ρ_2

  prerequisites -

  common part: YY'

  ρ_3 can be interpreted as a derived tripel by $X(YY')Z$.

Example 3.13

(Loop1) ABC, ACD, ADE, AEB $\Rightarrow$ ABE:

(See Diagram 3.4 (page 27).)

- $\rho_1 : A : \sigma$, $B : \sigma = \tau$, $C : \tau$

  generated by ABC from σ, τ

  prerequisite B

  common part B

- $\rho_2 : A : \rho_1 = \sigma$, $C : \rho_1 = \tau$, $D : \tau$

  generated by ACD from ρ_1, τ

  prerequisite -

  common part -

  ρ_2 cannot be interpreted as a derived tripel, as there was a prerequisite used in its derivation (B), but the common part in ρ_2 is $\emptyset$.

- ρ_3 similar to ρ_2 :

  $\rho_3 : A : \rho_2 = \sigma$, $D : \rho_2 = \tau$, $E : \tau$

  generated by ADE from ρ_2, τ

  prerequisite -

  common part -

  ρ_3 cannot be interpreted as a derived tripel, as there was a prerequisite used in its derivation (B), but the common part in ρ_3 is $\emptyset$.

- $\rho_4 : A : \rho_3 = \sigma$, $E : \rho_3 = \tau$, $B : \sigma = \tau$

  generated by AEB from ρ_3, τ

  prerequisites -

  common part B

  ρ_4 can be interpreted as the common part B contains all prerequisites used in its derivation. ABE is the only non-trivial derived tripel.

  Note that we could, e.g., also have replaced ACD by $AC'(DC'')$, where $C = C' \cup C''$, using rule (c), Weak Union.

Example 3.14

$BA(CD)$, $DF(CE)$, $(AB)(CD)(EF) \Rightarrow B(ADF)(CE)$:

(See Diagram 3.3 (page 26) lower part.)

This example shows that we may need an assumption in the interior of the tree (in the construction of ρ_3, we use $D : \sigma = \tau$).

- $\rho_1 : A : \sigma = \tau$, $B : \sigma$, $C : \tau$, $D : \tau$

  generated by $BA(CD)$ from σ, τ

  prerequisites A

  common part A

- $\rho_2 : C : \tau$, $D : \sigma$, $E : \tau$, $F : \sigma = \tau$

  generated by $DF(CE)$ from σ, τ

  prerequisite F

  common part F

- ρ_3 : A: $\rho_1 = \sigma = \tau$, $B : \rho_3 = \sigma$, $C : \rho_1 = \rho_2 = \tau$, $D : \rho_1 = \rho_2 = \sigma = \tau$, $E : \rho_2 = \tau$, $F : \rho_2 = \sigma = \tau$

  generated by $(AB)(CD)(EF)$ from ρ_1, ρ_2

  prerequisite D

  common part ADF

  So ρ_3 can be seen as the derived tripel $B(ADF)(CE)$ (but NOT as $(AB)(DF)(CE)$ etc., as DF does not contain ADF.

Example 3.15

$(AA')BC$, $AD(CD')$, $(AB')C(C'D)$, $(A'B')C(C'D')$, $(AD)(B'CC')(A'D')$, $BC(ADD') \Rightarrow A(BD)(CD')$:

(See Diagram 3.5 (page 28).)

This example shows that we may need an equality (here α and β in the construction of ρ_5) which is not related to σ and τ. Of course, we cannot use it as an assumption, but we know the equality by construction.

α and β will not be known, they are fixed, unknown fragments.

- $\rho_1 : A : \sigma$, $A' : \sigma$, $B : \sigma = \tau$, $B' : \alpha$, $C : \tau$

  generated by $(AA')BC$ from σ, τ

  prerequisites B

  common part B

- $\rho_2 : A : \sigma$, $C : \tau$, $C' : \beta$, $D : \sigma = \tau$, $D' : \tau$

  generated by $AD(CD')$ from σ and τ

  prerequisite D

  common part D

- $\rho_3 : A : \sigma$, $B' : \alpha$, $C : \tau$, $C' : \beta$, $D : \sigma = \tau$

  generated by $(AB')C(C'D)$ from ρ_1 and ρ_2

  prerequisite -

  common part D

- $\rho_4 : A' : \sigma$, $B' : \alpha$, $C : \tau$, $C' : \beta$, $D' : \tau$

  Generated by $(A'B')C(C'D')$ from ρ_1 and ρ_2

  prerequisites -

  common part -

- $\rho_5 : A : \sigma$, $A' : \sigma$, $B' : \alpha$, $C : \tau$, $C' : \beta$, $D : \tau$, $D' : \tau$

  generated by $(AD)(B'CC')(A'D')$ from ρ_3 and ρ_4

  prerequisites - (note that equality on B' and C' is by construction of ρ_3 and ρ_4, and not by a prerequisite on σ and τ)

  common part: D

- $\rho_6 : A : \sigma$, $B : \sigma = \tau$, $C : \tau$, $D : \sigma = \tau$, $D' : \tau$

  generated by $BC(ADD')$ from ρ_1 and ρ_5

  prerequisites -

  common part: BD

  Thus, ρ_6 may be seen as derived tripel $A(BD)(CD')$

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